

A Fresh Look at Coding for q -ary Symmetric Channels

Claudio Weidmann and Gottfried Lechner

Abstract

This paper studies coding schemes for the q -ary symmetric channel based on binary low-density parity-check (LDPC) codes that work for any alphabet size $q = 2^m$, $m \in \mathbb{N}$, thus complementing some recently proposed packet-based schemes requiring large q . First, theoretical optimality of a simple layered scheme is shown, then a practical coding scheme based on a simple modification of standard binary LDPC decoding is proposed. The decoder is derived once from first principles and once using a factor-graph representation of a front-end that maps q -ary symbols to groups of m bits connected to a binary code. The front-end can be processed with a complexity that is linear in $m = \log_2 q$. An extrinsic information transfer chart analysis is carried out and used for code optimization. Finally, it is shown how the same decoder structure can also be applied to a larger class of q -ary channels.

Index Terms

q -ary symmetric channel, low-density parity-check (LDPC) codes, decoder front-end.

I. INTRODUCTION

THE q -ary symmetric channel (q -SC) with error probability ϵ takes a q -ary symbol at its input and outputs either the unchanged input symbol, with probability $1 - \epsilon$, or one of the other $q - 1$ symbols, with probability $\epsilon/(q - 1)$. It has attracted some attention recently as a more general channel model for packet-based error correction. For very large q , its appropriately normalized capacity approaches that of an erasure (packet loss) channel. In the following, we will only consider channel alphabets of size $q = 2^m$ with $m \in \mathbb{N}$.

The capacity of the q -SC with error probability ϵ is

$$C_{q\text{-SC}} = m - h(\epsilon) - \epsilon \log_2(2^m - 1)$$

bits per channel use, where $h(x) = -x \log_2 x - (1-x) \log_2(1-x)$ is the binary entropy function. Asymptotically in m , the normalized capacity $C_{q\text{-SC}}/m$ thus approaches $1 - \epsilon$, which is the capacity of the binary erasure channel (BEC) with erasure probability ϵ .

Recent work [1]–[5] has shown that it is possible to approach $C_{q\text{-SC}}$ for large alphabet sizes $q = 2^m$, with symbols of hundreds to thousands of bits, and complexity $O(\log q)$ per code symbol. The focus of the present work is on smaller q , with symbols of tens of bits at most, although the presented coding techniques will work for any $q = 2^m$.

The q -ary channel input and output symbols will be represented by binary vectors of length m . Hence a simplistic coding approach consists in decomposing the q -SC into m binary symmetric channels (BSCs) with crossover probability

$$\epsilon_{\text{BSC}} = \frac{q\epsilon}{2(q-1)} = \frac{\epsilon}{2(1-2^{-m})}, \quad (1)$$

which have capacity $C_{\text{BSC}} = 1 - h(\epsilon_{\text{BSC}})$ each.

We briefly study the normalized capacity loss, $\Delta = C_{q\text{-SC}}/m - C_{\text{BSC}}$, which results from (wrongly) assuming that the q -SC is composed of independent BSCs. For fixed m , we have $\lim_{\epsilon \rightarrow 0} \Delta = 0$; so using binary codes with independent decoders on the m -fold BSC decomposition might be good enough for small ϵ (e.g. $\epsilon < 10^{-3}$). However, Figure 1 shows that the relative capacity loss $m\Delta/C_{q\text{-SC}} = 1 - mC_{\text{BSC}}/C_{q\text{-SC}}$ increases close to linearly in ϵ . For fixed ϵ , we have $\lim_{m \rightarrow \infty} \Delta = h(\epsilon/2) - \epsilon$, which can be a substantial fraction of the normalized q -SC capacity (e.g., for $\epsilon = 0.1$, $h(\epsilon/2) - \epsilon = 0.19$). Figure 2 shows that already for small m , the q -SC capacity is substantially larger than what can be achieved with the BSC decomposition. Clearly, there is a need for coding schemes targeted at “large” ϵ (say $\epsilon > 10^{-1}$), but moderate m (say $2 \leq m < 20$), which are not well handled by methods for large q . For example, when using a verification-based decoder and coding over blocks of n symbols, the symbols should have $m = \Omega(n)$ bits [1].

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Claudio Weidmann is with INTHFT, Vienna University of Technology, Vienna, Austria, and LSS-CNRS-Supélec, Gif-sur-Yvette, France.

Gottfried Lechner is with the Institute for Telecommunications Research, University of South Australia, Mawson Lakes Boulevard, Mawson Lakes 5095, South Australia, Australia. This work has been supported by the Australian Research Council under ARC grant DP0881160.

E-mail: claudio.weidmann@ieee.org, gottfried.lechner@unisa.edu.au

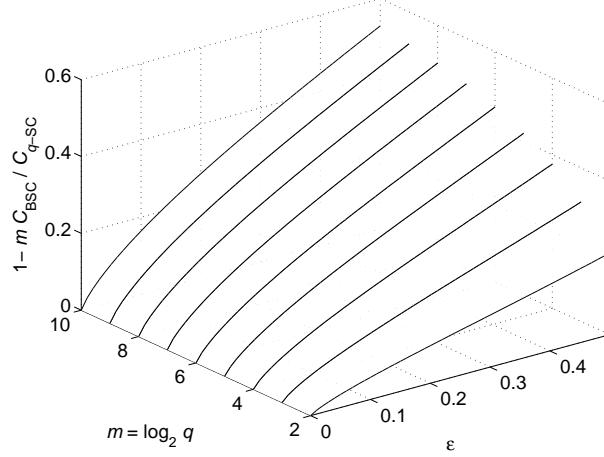


Fig. 1. Relative capacity loss $1 - \frac{mC_{\text{BSC}}}{C_{q-\text{SC}}}$ of marginal BSC vs. q -SC.

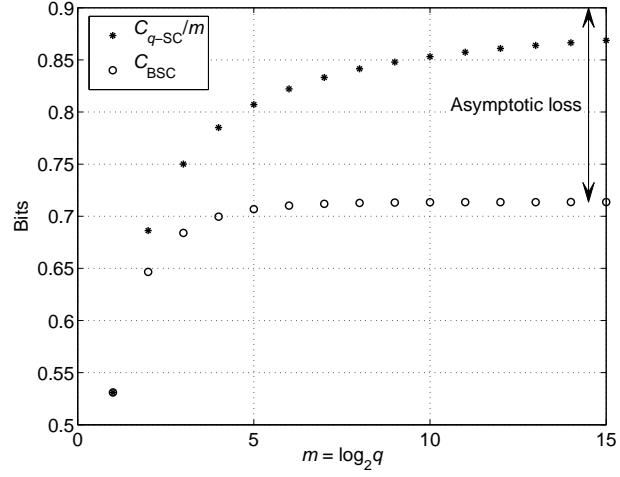


Fig. 2. Capacity of q -SC vs. marginal BSC (error probability $\epsilon = 0.1$).

One example application, which also motivated this work, is Slepian-Wolf coding of q -ary sources with a discrete impulse-noise correlation model, using q -SC channel code syndromes as fixed-rate (block) source code. This can then be used as a building block for a Wyner-Ziv coding scheme, where q is the number of scalar quantizer levels, or for fixed-rate quantization of sparse signals with a Bernoulli- ϵ prior on being nonzero [6]. Clearly, q will be only moderately large in such a scenario.

The capacity loss of the binary decomposition is due to the fact that the correlation between errors on the binary sub-channels is not taken into account, i.e., since an error on one sub-channel of the q -SC implies a symbol error, it will be more likely for the other sub-channels to be in error as well. A better approach would be the use of non-binary low-density parity-check (LDPC) codes over $\text{GF}(q)$ [7]. While that would take into account the dependency between bit errors within a symbol, the decoding complexity of the associated non-binary LDPC decoder is $O(q \cdot \log q)$ or at least $O(q)$ when using sub-optimal algorithms [8].

Instead, this work focuses on a modified binary LDPC decoder of complexity $O(\log q)$. Section II studies an ideal scheme using layers of different-rate binary codes, providing the key intuition that once a bit error is detected, the remaining bits of the symbol may be treated as erasures without loss in rate. Section III then proposes a scheme using a single binary code and develops the new variable node decoding rules from first principles, by factorizing the posterior probabilities. Section IV shows that the new decoding rule is equivalent to a front-end that maps q -ary symbols to groups of m bits and studies its factor graph representation. Section V provides the extrinsic information transfer chart characterization of this q -SC front-end, which is used in Section VI to design optimized LDPC codes. Finally, Section VII shows that these decoding methods extend to a larger class of q -ary channels.

II. LAYERED CODING SCHEME

We study the following layered coding scheme based on binary codes. Blocks of k symbols $[u^1, u^2, \dots, u^k]$ are split into m bit layers $[u_i^1, u_i^2, \dots, u_i^k]$, $i = 1, \dots, m$, and each layer is independently encoded with a code for a binary symmetric erasure

channel (BSEC) with erasure probability δ_i and crossover probability ϵ_i , to be specified below. The channel input symbol will be denoted $x^j = [x_1^j, x_2^j, \dots, x_m^j]^\top$, where x_i^j is the i -th bit of the j -th symbol, while the corresponding channel output is $y^j = [y_1^j, y_2^j, \dots, y_m^j]^\top$, respectively y_i^j .

The key idea is to decode the layers in a fixed order and to declare *bit erasures* at those symbol positions in which a bit error occurred in a previously decoded layer. This saves on the code redundancy needed in the later layers, since erasures can be corrected with less redundancy than bit errors.

The decoder performs successive decoding of the m layers, starting from layer 1. All errors corrected at layer i and below are forwarded to layer $i+1$ as erasures, that is all bit error positions found in layers 1 up to i will be marked as erased in layer $i+1$, even though the channel provides a (possibly correct) binary output for those positions. Let ϵ_i be the probability that the channel output y is equal to the input x in bit positions 1 to $i-1$ and differs in position i (i.e. $[y_1, y_2, \dots, y_{i-1}]^\top = [x_1, x_2, \dots, x_{i-1}]^\top$ and $y_i \neq x_i$). A simple counting argument shows that there are 2^{m-i} such binary vectors $y \neq x$, out of a total $2^m - 1$. The i -th binary sub-channel is thus characterized by

$$\epsilon_i = \frac{2^{m-i}}{2^m - 1} \epsilon, \quad (2)$$

$$\delta_i = \sum_{j=1}^{i-1} \epsilon_j = \frac{2^m - 2^{m-i+1}}{2^m - 1} \epsilon. \quad (3)$$

Theorem 1 *The layered scheme achieves q -SC capacity.*

Proof: We assume an ideal scheme, in which all layers operate at their respective BSEC capacities and correct all errors and erasures. The $BSEC(\delta_i, \epsilon_i)$ capacity is

$$C_{\text{BSEC}} = (1 - \delta_i) \left(1 - h \left(\frac{\epsilon_i}{1 - \delta_i} \right) \right). \quad (4)$$

Hence the sum of the layer rates becomes

$$\sum_{i=1}^m R_i = \sum_{i=1}^m (1 - \delta_i) \left(1 - h \left(\frac{\epsilon_i}{1 - \delta_i} \right) \right) \quad (5)$$

$$= m + \sum_{i=1}^m \left\{ -\delta_i - (1 - \delta_i) \log_2(1 - \delta_i) + \epsilon_i \log_2 \epsilon_i + (1 - \delta_{i+1}) \log_2(1 - \delta_{i+1}) \right\} \quad (6)$$

$$= m + \sum_{i=1}^m \left\{ -(m-i)\epsilon_i + \epsilon_i \log_2 \epsilon_i \right\} + (1-p) \log_2(1-p) \quad (7)$$

$$= m + \sum_{i=1}^m \left\{ -(m-i)\epsilon_i + (m-i)\epsilon_i \right\} + p \log_2 p - p \log_2(2^m - 1) + (1-p) \log_2(1-p) \quad (8)$$

$$= m - h(p) - p \log_2(2^m - 1) = C_{q\text{-SC}},$$

where (5) follows from (4) and the definition of the layered scheme, (6) follows from $\delta_i + \epsilon_i = \delta_{i+1}$ (which holds up to $i = m$, when $\delta_{m+1} = \epsilon$), (7) follows from the evaluation of the telescoping sum and $\sum_{i=1}^m \delta_i = \sum_{i=1}^m (m-i)\epsilon_i$, and (8) follows from substituting (2) for ϵ_i . ■

The intuition behind the optimality of this (seemingly suboptimal) layered scheme is that once a bit error (and thus a symbol error) has been detected, all the following layers have bit error probability 1/2 in that position, since the q -SC assigns uniform probabilities over the possible symbol error values. Now the BSC(1/2) has zero capacity and so the concerned bits can be treated as erasures with no loss.

Since the capacity gain relative to $m-1$ becomes smaller with increasing m , as can be seen in Figure 2, an interesting variant of the layered scheme is to use $\mu < m$ BSEC layers as above, followed by a single “thicker” layer, which sends the remaining $m-\mu$ bits (per symbol) over the same $BSEC(\delta_{\mu+1}, \epsilon_{\mu+1})$. In particular, this could even be beneficial in practical implementations, since combining layers leads to longer codewords and thus better codes, which might outweigh the theoretical capacity loss. The next section will show that it is actually possible to reap the benefits of a single large binary code, without layers, by using a decoder that exploits the dependencies among the bits in a symbol.

III. BIT-SYMMETRIC CODING SCHEME

The main disadvantage of the layered scheme is that in a practical implementation, each layer will need a different code that is tuned to the effective erasure and error probabilities resulting from the layers preceding it. This makes it impractical for hardware implementation, since the required silicon area would necessarily grow with the number of layers.

Ideally, a coding scheme for the q -SC should be symmetric in the bits composing a symbol, that is, no artificial hierarchy among bit layers should be introduced (notice that the order of the bit layers may be chosen arbitrarily). We propose to encode all bits composing the symbols with one “big” binary code, which needs to satisfy just slightly stricter constraints than an ordinary code for the BSC, while the decoder alone will exploit the knowledge about the underlying q -SC. The key concept that should carry over from the layered scheme is that the decoder is able to declare erasures at certain symbol positions and thus needs less error correction capability (for part of the bits in erased symbols).

Our proposal for a practical symmetric q -SC coding scheme relies on a LDPC code with information block size $K = mk$ bits and channel block size $N = mn$ bits. We assume that the variable nodes (VNs) in the decoder receive independent extrinsic soft estimates of X_i^j (that is, bit i of code symbol/vector X^j , for $i = 1, \dots, m$, $j = 1, \dots, n$) from the check nodes (CNs). These amount to estimates of $P(X_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]})$ or the corresponding log-likelihood ratio (LLR), $L(X) = \log(\Pr(X=0)/\Pr(X=1))$. (As usual, the notation $\mathbf{y}^{[j]}$ denotes the block consisting of all symbols/vectors except the j -th.) In particular, the extrinsic estimate of X_i^j is assumed to be independent of the other bits $X_{[i]}^j$ of the same symbol, so that we may write

$$P(X_i^j | X_1^j, \dots, X_{i-1}^j, \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}) = P(X_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}).$$

In the standard case, these independence assumptions are justified by the fact that asymptotically in the block length, the neighborhood of a VN in the LDPC decoder computation graph becomes a tree [9, Chap. 3]. Unfortunately, the q -SC VN message computation rule has to depend on the other bits in the same symbol, in order to account for the bit error correlation, and thus will introduce cycles. However, this problem can be alleviated by imposing an additional constraint on the code, namely that the parity checks containing X_i^j do not involve any of the bits $X_{[i]}^j$. This is a necessary condition for the above intra-symbol independence assumption and may be achieved by using an appropriate edge interleaver in the LDPC construction. Then the cycles introduced by the VN message rule will grow asymptotically and are thus not expected to lead to problems in practice.

As suggested above, the properties of the q -SC can be taken into account via a simple modification of the VN computation in the message-passing decoding algorithm for binary LDPC codes. We factor the *a posteriori* probability (APP) of symbol X^j as follows:

$$\begin{aligned} P(X^j | \mathbf{Y} = \mathbf{y}) &\doteq P(Y^j = y^j | X^j) P(X^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}) \\ &= P(Y^j = y^j | X^j) \prod_{i=1}^m P(X_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}), \end{aligned} \quad (9)$$

where the factorization in (9) is made possible by the above independence assumption (the symbol \doteq denotes equality up to a positive normalization constant). Using the definition of the q -SC, this becomes

$$P(X^j = x^j | \mathbf{Y} = \mathbf{y}) \doteq \begin{cases} (1-p) \prod_{i=1}^m P(X_i^j = x_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}), & x^j = y^j, \\ \frac{p}{q-1} \prod_{i=1}^m P(X_i^j = x_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}), & x^j \neq y^j. \end{cases} \quad (10)$$

We define the extrinsic probability that $X^j = y^j$ as

$$\beta^j = \prod_{i=1}^m P(X_i^j = y_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]}) = \prod_{i=1}^m p_i^j,$$

where we introduced

$$p_i^j = P(X_i^j = y_i^j | \mathbf{Y}^{[j]} = \mathbf{y}^{[j]})$$

for notational convenience. The normalization constant in (10) thus becomes

$$\gamma^j = (1-\epsilon)\beta^j + \frac{\epsilon}{q-1}(1-\beta^j).$$

Then the bit APP may be obtained by the marginalization

$$P(X_i^j = x_i^j | \mathbf{Y} = \mathbf{y}) = \sum_{x_{[i]}^j \in \{0,1\}^{m-1}} P(X^j = x^j | \mathbf{Y} = \mathbf{y}), \quad (11)$$

which may be written as

$$P(X_i^j = x_i^j | \mathbf{Y} = \mathbf{y}) = \begin{cases} \left[(1-\epsilon)\beta^j + \frac{\epsilon}{q-1}(1-\beta^j) \right] \cdot \frac{p_i^j}{\gamma^j}, & x_i^j = y_i^j, \\ \frac{\epsilon}{q-1} \cdot \frac{1-p_i^j}{\gamma^j}, & x_i^j \neq y_i^j, \end{cases} \quad (12)$$

where $\beta_{[i]}^j = \beta^j / p_i^j$ is the “intra-symbol” extrinsic probability that $X^j = y^j$, using no information on bit X_i^j . Finally, we may express the *a posteriori* bit-level LLR as

$$\begin{aligned} L_{\text{app}}(X_i^j = y_i^j) &= \log \left(\frac{(q-1)\beta_j - q\epsilon\beta^j + \epsilon p_i^j}{\epsilon(1-p_i^j)} \right) \\ &= \log \left(\frac{(q-1)\beta_{[i]}^j - q\epsilon\beta_{[i]}^j + \epsilon}{\epsilon} \cdot \frac{p_i^j}{1-p_i^j} \right) \\ &= \underbrace{\log \left(1 + \frac{q-q\epsilon-1}{\epsilon} \cdot \beta_{[i]}^j \right)}_{L_{\text{ch}}(X_i^j = y_i^j)} + L_{\text{extr}}(X_i^j = y_i^j). \end{aligned} \quad (13)$$

The usual $L(X)$ is obtained from $L(X=y) = \log(\Pr(X=y)/\Pr(X=1-y))$ via a sign flip, $L(X) = (1-2y)L(X=y)$.

The second term in (13) corresponds to the extrinsic information from the CNs that is processed at the VNs in order to compute the bit APP in standard binary LDPC decoding. The difference lies in the first term in (13), which corresponds to the channel LLR (which would be $L_{\text{ch}}(X) = \log(P(y|X=0)/P(y|X=1))$ in the binary case). When the extrinsic information on the bits $X_{[i]}^j$ favors the hypothesis $X^j \neq y^j$, the product $\beta_{[i]}^j$ will be small and therefore L_{ch} in (13) will be close to zero, which is equivalent to declaring a bit erasure. This shows that the symmetric LDPC scheme relies on “distributed” soft bit erasure estimates, while in the layered scheme the erasures are declared in a *hard* “top-down” fashion.

Equation (13) describes the modification of the VN computation that turns a message-passing binary LDPC decoder into one for the q -ary symmetric channel. The outgoing VN messages are computed as usual by subtracting the incoming edge message from $L_{\text{app}}(X_i^j)$; also the CN messages are the same as in the binary case. For practical implementation purposes, (13) should probably be modified (approximated) in order to avoid switching back and forth between probabilities and LLRs when computing β^j . A final detail is the specification of the initial channel LLR $L_{\text{ch}}^{(0)}$ in (13), which is needed to start the decoder iterations. By inserting the memoryless worst-case estimate $\beta_{[i]}^j = 2^{-m+1}$ into (13), we obtain

$$L_{\text{ch}}^{(0)} = \log \left(\frac{2(1-2^{-m}) - \epsilon}{\epsilon} \right), \quad (14)$$

which is exactly the channel LLR for the marginal BSC with crossover probability ϵ_{BSC} given in (1).

Notice that the decoder iterations are exclusively between VN (13) and CN computations, like in the binary case. However, computing (13) at the VNs requires the extrinsic information (the CN messages) for all bits within a symbol; this might be considered an additional level of message exchanges (specifically, plain copying of messages), but it does not involve iterations of any kind. The complexity increase compared to binary LDPC decoding is on the order of at most m operations per variable node, depending on the scheduling. In fact, the marginalization (11) is reminiscent of a combined detector and VN decoder for binary LDPC codes that are directly mapped to larger signal constellations [10]. Thanks to the symmetry of the q -SC, here it is not necessary to actually sum over all q symbol values. Other similar work includes iterative demapping and decoding [11], [12], which however involves proper iterations between the demapper and the LDPC decoder, being treated as separate functional blocks. The next section will take that point of view.

IV. FACTOR GRAPH REPRESENTATION OF A q -SC FRONT-END

This section presents a different view on the bit-symmetric coding scheme by considering a q -SC front-end for an LDPC decoder, similar to approaches for iterative demapping and decoding [11], [12]. The front-end takes into account the correlation between the errors on the m bit layers. Its factor graph representation allows to formulate a message passing algorithm that computes essentially the same quantities as the “direct” algorithm in Sec. III, but displays more clearly the opportunity for further complexity reduction by appropriate message scheduling.

As before, let \mathbf{x} denote the vector of q -ary channel input symbols x^j ($j = 1, \dots, n$) and let x_i^j denote bit i of symbol j . In the same way, the output of the channel is represented by \mathbf{y} , y^j and y_i^j , and the errors are denoted by \mathbf{e} , e^j and e_i^j , respectively. We assume w.l.o.g. that $e_i^j = x_i^j \oplus y_i^j$, where \oplus denotes addition in GF(2), and $e^j = x^j \oplus y^j$, by extension.

Let $\chi_C(\mathbf{x})$ be the characteristic function of the code, which evaluates to one if \mathbf{x} is a codeword and to zero otherwise. Furthermore assuming that the transmitted codewords are equally likely, the *a posteriori* probability satisfies the proportionality relation

$$f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) \doteq \chi_C(\mathbf{x}) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}).$$

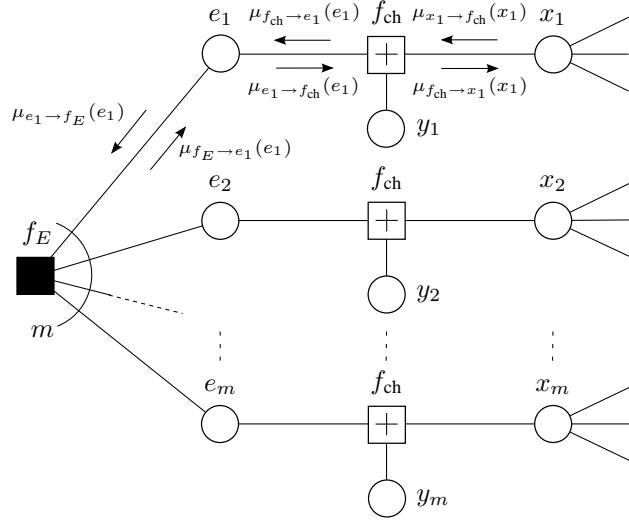


Fig. 3. Factor graph representation of the q -SC front-end for symbol x^j (the symbol index j is omitted).

The q -ary channel is assumed to be memoryless (on the symbol level), leading to a factorization of $f_{\mathbf{Y}|\mathbf{X}}$ as

$$\begin{aligned} f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) &\doteq \chi_C(\mathbf{x}) \prod_{j=1}^n f_{Y|X}(y^j|x^j) \\ &= \chi_C(\mathbf{x}) \prod_{j=1}^n \left\{ \sum_{e^j} f_{Y,E|X}(y^j, e^j | x^j) \right\} \\ &= \chi_C(\mathbf{x}) \prod_{j=1}^n f_{Y,E|X}(y^j, x^j \oplus y^j | x^j), \end{aligned}$$

where in the last step we used the fact that x^j , e^j and y^j are related in a deterministic way.

Given an error symbol e^j , the bit-layers are independent of each other, leading to

$$\begin{aligned} f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) &\doteq \chi_C(\mathbf{x}) \prod_{j=1}^n \left\{ f_E(e^j) f_{Y|X,E}(y^j | x^j, e^j) \right\} \\ &= \chi_C(\mathbf{x}) \prod_{j=1}^n \left\{ f_E(e^j) \prod_{i=1}^m f_{\text{ch}}^j(y_i^j | x_i^j, e_i^j) \right\}, \end{aligned}$$

where $f_{\text{ch}}^j = f_{Y_i^j | X_i^j, E_i^j}$.

This factorization is shown for one symbol x^j in Fig. 3, where the edges on the right side are connected to the check nodes of the LDPC code, i.e. the factorization of the characteristic function $\chi_C(\mathbf{x})$. In the following, we will denote a message sent from a variable node x to a function node f and vice versa as $\mu_{x \rightarrow f}(x)$ and $\mu_{f \rightarrow x}(x)$, respectively [13].

A. Message Passing Rules

We apply the sum-product algorithm to compute the marginals $f_{X^j|\mathbf{Y}}(x_i^j|\mathbf{y})$ for all symbols j and all bits i , i.e. to perform *a posteriori* decoding for every bit of the transmitted codeword. In the following we will describe the operations for a single q -ary symbol and therefore omit the symbol index j for convenience.

First, we define the local functions and derive the message passing rules for the function nodes f_{ch} and f_E . Since the variable nodes e_i have degree two, they just forward the incoming messages, e.g. $\mu_{e_1 \rightarrow f_{\text{ch}}}(e_1) = \mu_{f_E \rightarrow e_1}(e_1)$ and $\mu_{e_1 \rightarrow f_E}(e_1) = \mu_{f_{\text{ch}} \rightarrow e_1}(e_1)$.

1) Bit Layer Channels: In the binary sub-channels, x_i , y_i and e_i have to sum up to zero in GF(2), therefore the local functions f_{ch} are given by

$$f_{\text{ch}}(y_i | x_i, e_i) = [y_i = x_i \oplus e_i],$$

where $[P]$ evaluates to 1 if the expression P is true and to 0 otherwise.

Having defined the local function, we can derive the message passing algorithm by applying the rules of the sum-product algorithm. Given a message $\mu_{x_i \rightarrow f_{\text{ch}}}(x_i)$ and a received bit y_i , the channel function node computes

$$\begin{aligned}\mu_{f_{\text{ch}} \rightarrow e_i}(e_i) &= \sum_{\sim\{e_i\}} \left(f_{\text{ch}}(y_i|x_i, e_i) \cdot \mu_{x_i \rightarrow f_{\text{ch}}}(x_i) \right) \\ &= \mu_{x_i \rightarrow f_{\text{ch}}}(e_i \oplus y_i),\end{aligned}$$

where $\sum_{\sim\{a\}}$ denotes the marginalization over all variables except a . In the same way, the message sent back to variable node x_i is computed as

$$\begin{aligned}\mu_{f_{\text{ch}} \rightarrow x_i}(x_i) &= \sum_{\sim\{x_i\}} \left(f_{\text{ch}}(y_i|x_i, e_i) \cdot \mu_{e_i \rightarrow f_{\text{ch}}}(e_i) \right) \\ &= \mu_{e_i \rightarrow f_{\text{ch}}}(x_i \oplus y_i).\end{aligned}$$

2) *Error Patterns:* The function node f_E represents the probability that a certain binary error pattern occurs. In the q -ary symmetric channel, every error pattern has the same probability $\frac{\epsilon}{q-1}$, except the all-zero pattern that has probability $1-\epsilon$, thus

$$f_E(e_1, \dots, e_m) = \begin{cases} 1-\epsilon, & \text{if } e_1 = \dots = e_m = 0 \\ \frac{\epsilon}{q-1}, & \text{otherwise.} \end{cases}$$

The derivation of the outgoing messages of f_E leads to

$$\begin{aligned}\mu_{f_E \rightarrow e_i}(e_i) &= \sum_{\sim\{e_i\}} \left(f_E(e_1, \dots, e_m) \cdot \prod_{i' \neq i} \mu_{e_{i'} \rightarrow f_E}(e_{i'}) \right) \\ &= \begin{cases} (1-p) \cdot \beta_{[i]} + \frac{\epsilon}{q-1} (1 - \beta_{[i]}) & \text{if } e_i = 0 \\ \frac{\epsilon}{q-1} & \text{if } e_i = 1,\end{cases}\end{aligned}$$

where $\beta_{[i]}$ is defined as

$$\beta_{[i]} = \prod_{i' \neq i} \mu_{e_{i'} \rightarrow f_E}(0). \quad (15)$$

B. Simplification of the Messages

Since all involved variables are binary, we can represent the messages by scalar quantities. For the messages from and to the variable nodes x_i^j of the LDPC code, we use log-likelihood ratios of the corresponding messages

$$\begin{aligned}L_{a,i} &= \log \frac{\mu_{x_i \rightarrow f_{\text{ch}}}(0)}{\mu_{x_i \rightarrow f_{\text{ch}}}(1)}, \\ L_{\text{ch},i} &= \log \frac{\mu_{f_{\text{ch}} \rightarrow x_i}(0)}{\mu_{f_{\text{ch}} \rightarrow x_i}(1)}.\end{aligned}$$

The messages from the function nodes f_{ch} to the node f_E are defined as the probability of no error

$$p_i = \mu_{f_{\text{ch}} \rightarrow e_i}(0) = \mu_{e_i \rightarrow f_E}(0),$$

and the messages from the function node f_E are defined as the error probability

$$\epsilon_i = \frac{\mu_{f_E \rightarrow e_i}(1)}{\mu_{f_E \rightarrow e_i}(0) + \mu_{f_E \rightarrow e_i}(1)}.$$

Using these definitions, we can reformulate the computation rules at the function nodes as

$$p_i = \frac{e^{\frac{L_{a,i}}{2} \cdot (1-2y_i)}}{e^{\frac{L_{a,i}}{2}} + e^{-\frac{L_{a,i}}{2}}},$$

and

$$\begin{aligned}\epsilon_i &= \frac{\frac{\epsilon}{q-1}}{(1-\epsilon) \cdot \beta_{[i]} + \frac{\epsilon}{q-1} (1 - \beta_{[i]}) + \frac{\epsilon}{q-1}} \\ &= \frac{\epsilon}{2\epsilon + \beta_{[i]}(q - \epsilon q - 1)}\end{aligned} \quad (16)$$

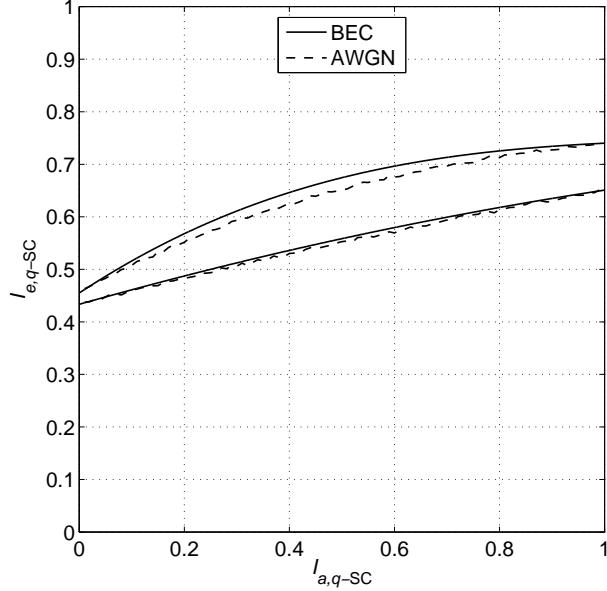


Fig. 4. EXIT function for the q -SC front-end with BEC (analytical) and Gaussian (simulated) *a priori* messages for $\epsilon = 0.25$, $m = 4$ (lower curves) and $m = 8$ (upper curves).

For the initialization, we set all *a-priori* L-values $L_{a,i}$ to zero, leading to $\beta_{[i]} = 2^{-(m-1)} = 2/q$. Inserting this in (16) leads to

$$\epsilon_i = \frac{\epsilon q}{2(q-1)} = \epsilon_{\text{BSC}}.$$

Finally, the messages sent from the function nodes f_{ch} to the variable nodes x_i are computed as

$$L_{\text{ch},i} = (1 - 2y_i) \cdot \log \frac{1 - \epsilon_i}{\epsilon_i}.$$

C. Complexity

Clearly, the processing complexity of the front-end is dominated by (15) and is thus $O(\log q)$ per symbol, as in verification decoding [1]. This means that the overall decoding complexity scales linearly in the number of transmitted bits, independently of the symbol alphabet size q . Complexity may be further reduced by a constant factor (i.e. *not* its order) if the front-end messages are not recomputed on every LDPC decoder iteration; this is often sufficient in practice.

V. EXIT ANALYSIS OF THE q -SC FRONT-END

In this section, we analyze the q -SC front-end using extrinsic information transfer (EXIT) charts [14], which will later also be used for the design (optimization) of LDPC codes.

Let the *a priori* and *extrinsic* messages for the front-end denote the messages between the bit layer sub-channels f_{ch} and the bit nodes x_i of the LDPC code, that is, $\mu_{x_i \rightarrow f_{\text{ch}}}(x_i)$ and $\mu_{f_{\text{ch}} \rightarrow x_i}(x_i)$, respectively. For the EXIT chart analysis, the front-end is characterized by the transfer function $I_e(I_a)$, where I_a and I_e denote the *a priori* and the *extrinsic* mutual information, respectively, between the messages and the corresponding bits.

First, we derive the EXIT function of the front-end for the case when the *a priori* messages are modeled as coming from a binary erasure channel (BEC). In this special case, the EXIT functions have important properties (e.g. the area-property), which were shown in [14].

In the case of a BEC, the variables x_i are either known to be error-free or erased, thus simplifying the computation of $\beta_{[i]}$ defined in (15) as follows. The product in (15) has $m-1$ factors, taking on the value $1/2$ if the bit x_i is erased, 0 if the bit x_i is not in agreement with the received bit y_i and 1 otherwise. If at least one message is not in agreement with the received bit, a symbol error occurred and $\beta_{[i]} = 0$. Otherwise, $\beta_{[i]} = 2^{-t}$, where t denotes the number of erased messages.

Since $\beta_{[i]} = 0$ corresponds to $\epsilon_i = 1/2$ through (16), this case is equivalently described by an erasure. Therefore, the extrinsic messages can be modeled as being transmitted over a binary symmetric erasure channel (BSEC), with parameters ϵ_{m-t} and δ_{m-t} given by (2) and (3), respectively, since t erasures correspond to the situation in layer $i = m-t$ in the layered scheme of Section II. Its capacity is thus

$$I_t = (1 - \delta_{m-t}) \left(1 - h \left(\frac{\epsilon_{m-t}}{1 - \delta_{m-t}} \right) \right). \quad (17)$$

In order to compute the extrinsic information $I_e(I_a)$ at the output of the front-end as

$$I_e(I_a) = \sum_{t=0}^{m-1} I_t \lambda_t(I_a), \quad (18)$$

we have to compute the probability $\lambda_t(I_a)$ that t messages are erased given *a priori* mutual information I_a . Let $\Delta_a = 1 - I_a$ denote the probability that an *a priori* message is erased. The probability that $t = 0, \dots, m-1$ out of $m-1$ messages are erased is computed as

$$\begin{aligned} \lambda_t(I_a) &= \binom{m-1}{t} \Delta_a^t (1 - \Delta_a)^{m-1-t} \\ &= \binom{m-1}{t} (1 - I_a)^t I_a^{m-1-t}. \end{aligned} \quad (19)$$

The EXIT function for BEC *a priori* messages can therefore be computed using (2), (3), (17) and (19) in (18).

According to the area theorem of EXIT charts [14], the capacity of each bit layer is given by the area below the EXIT function. Therefore, the capacity of the overall channel is given by

$$\begin{aligned} C_{\text{tot}} &= m \int_0^1 \sum_{t=0}^{m-1} I_t \lambda_t(I_a) dI_a \\ &= m \sum_{t=0}^{m-1} I_t \int_0^1 \lambda_t(I_a) dI_a \\ &= m \sum_{t=0}^{m-1} I_t \binom{m-1}{t} \int_0^1 (1 - I_a)^t I_a^{m-1-t} dI_a \\ &= m \sum_{t=0}^{m-1} I_t \frac{(m-1)!}{(m-1-t)! t!} \frac{t!(m-1-t)!}{m!} \\ &= \sum_{t=0}^{m-1} I_t = C_{q\text{-SC}}, \end{aligned} \quad (20)$$

where we used the fact that the integral corresponds to the definition of the beta function. The last sum in (20) is indeed equal to the capacity of the q -SC, as was already shown in the proof of Theorem 1.

Without iterative processing, the maximum achievable rate is given by $I_e(0)$. In that case, all *a priori* messages are erased and one has

$$I_{m-1} = 1 - h(\epsilon_1) = 1 - h\left(\frac{pq}{2(q-1)}\right) = C_{\text{BSC}},$$

which corresponds to the simplistic coding approach where the q -SC is decomposed into m BSCs.

Figure 4 shows an EXIT chart for BEC *a priori* messages. The value at $I_a = 0$ denotes the maximum achievable rate without iterative processing and decoding and the area below the EXIT function corresponds to the capacity of the q -SC. When using this front-end with an LDPC code, the *a priori* messages can not be modeled as coming from a BEC channel. For code design, we will make the approximation that the messages are Gaussian distributed, which is a common assumption when using EXIT charts. The simulated EXIT function using Gaussian *a priori* messages is also shown in Figure 4.

VI. LDPC DESIGN AND CONSTRUCTION

After obtaining the EXIT function of the q -SC front-end, we can design an LDPC code. Code design is an optimization problem that selects the degree distributions of the code in order to maximize the rate, under the constraint that the EXIT function of the overall LDPC code does not intersect the EXIT function of the q -SC front-end. Joint optimization of the variable and check node degree distributions is a nonlinear problem, but it was shown in [12] that the optimization of the check node degree distribution given a fixed variable node degree distribution is a linear programming problem, which can be solved efficiently.

For the optimization in [12], it was assumed that the decoder is allowed sufficient iterations between variable and check nodes before performing an outer iteration (i.e. processing the front-end). While this assumption is necessary for the optimization of the code, it can be dropped when running the actual decoder if the graph is free of (short) cycles, since the choice of scheduling in a cycle-free graph has no influence on the final result.

Since we want the variable nodes to have the same convergence behavior, we use variable-regular codes and fix the variable node degree to $d_v = 3$. Furthermore, we set the maximum check node degree to $d_{c,\max} = 50$ in order to obtain results for which codes with practical lengths can be constructed. The resulting optimized check node degree distribution [9, Chap. 3] has

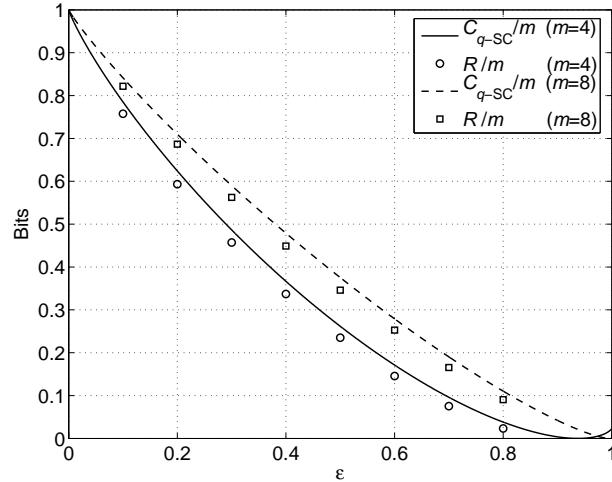


Fig. 5. Rate of optimized codes versus normalized capacity of the q -SC for $m = 4$ and $m = 8$.

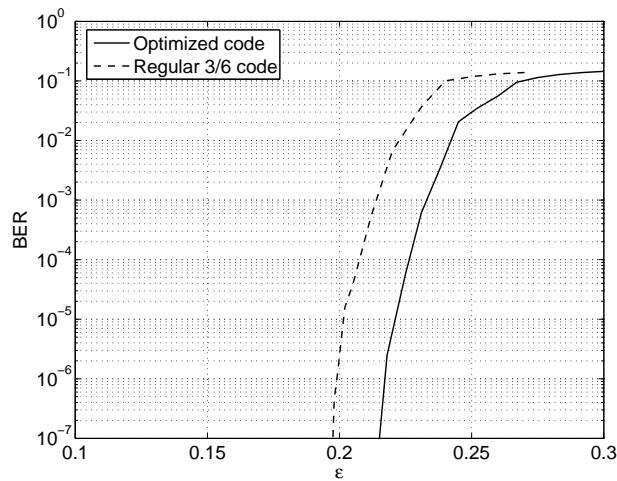


Fig. 6. Bit error rate simulation for rate 1/2 LDPC decoding ($m = 4$, $k = 1500$ symbols).

only few non-zero values in general, e.g. the optimized distribution for $m = 4$ and rate $R = 1/2$ is $\rho_4 = 0.1087$, $\rho_5 = 0.6753$, $\rho_{49} = 0.0299$ and $\rho_{50} = 0.1861$.

Fig. 5 shows the normalized capacity of the q -SC and the obtained optimized code rates versus the error probability ϵ of the q -SC for $m = 4$ and $m = 8$. Using this code optimization, we are able to design codes that perform close to capacity over a wide range of error probabilities for moderate q .

For the actual code construction, we used a modified version of the PEG algorithm [15], [16], which allows us to construct variable-regular codes with a given check node degree distribution. In order to avoid (short) cycles in the overall factor graph, one has to ensure that the bits of a given q -ary symbol do not participate in the same parity check. This can be achieved by using an appropriate interleaver between the q -SC front-end and the LDPC decoder.

To verify our derivations, we performed bit error rate simulations for an optimized binary LDPC code of rate $R = 1/2$ and length $N = 12000$, shown in Fig. 6. This code is optimized for a q -SC with $m = 4$, thus $N = 12000$ bits correspond to $k = 1500$ q -ary symbols. It has a decoding threshold of $\epsilon = 0.26$ (the Shannon limit for $R = 1/2$ is at $\epsilon = 0.29$). As a comparison we also show the bit error rate for a regular LDPC code with $d_v = 3$ and $d_c = 6$.

VII. GENERALIZATION TO CONDITIONALLY INDEPENDENT BINARY SUB-CHANNELS

The decoder in Section III is seen to easily extend to q -ary channels with modulo-additive noise, where the m bits of the noise (error pattern) are independent if an error occurs, i.e. if at least one bit is nonzero. This assumption leads to define the

$$\begin{aligned}
L_{\text{app}}(X_i^j = y_i^j) &= \log \left(\frac{(1-\epsilon)\beta^j + \alpha(1-\epsilon_{i|*})p_i^j \left[\prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j) - \prod_{k \neq i} (1-\epsilon_{k|*})p_k^j \right]}{\alpha\epsilon_{i|*}(1-p_i^j) \prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j)} \right) \\
&= \log \left(\frac{(1-\alpha)\beta_{[i]}^j + \alpha(1-\epsilon_{i|*}) \prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j)}{\alpha\epsilon_{i|*} \prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j)} \right) + \log \left(\frac{p_i^j}{1-p_i^j} \right) \\
&= \underbrace{\log \left(\frac{1-\epsilon_{i|*}}{\epsilon_{i|*}} + \frac{(1-\alpha)\beta_{[i]}^j}{\alpha\epsilon_{i|*} \prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j)} \right)}_{L_{\text{ch}}(X_i^j = y_i^j)} + L_{\text{extr}}(X_i^j = y_i^j). \tag{21}
\end{aligned}$$

q -ary symmetric channel with conditionally independent binary sub-channels (q -SC*), with transition probabilities

$$P_{Y|X}(y|x) = \begin{cases} 1-\epsilon, & y=x, \\ \alpha \prod_{i=1}^m \epsilon_{i|*}^{x_i \oplus y_i} (1-\epsilon_{i|*})^{1 \oplus x_i \oplus y_i}, & y \neq x, \end{cases}$$

where $\alpha = \frac{\epsilon}{1 - \prod_{i=1}^m (1-\epsilon_{i|*})}$

and \oplus denotes GF(2) addition (this definition encompasses all channels satisfying the above conditional independence assumption, from which it may be derived axiomatically). The defining parameters are the symbol error probability ϵ and the m conditional probabilities $\epsilon_{i|*}$, which are the bit error probabilities conditioned on the event that at least one of the other $m-1$ bits is in error. The marginal bit error probabilities will be $\epsilon_i = \alpha\epsilon_{i|*}$. Since the q -SC* is strongly symmetric, the uniform input distribution achieves its capacity

$$C_{q\text{-SC}*} = m - \alpha \sum_{i=1}^m h(\epsilon_{i|*}) + (1-\epsilon) \log_2(1-\epsilon) + \alpha \log_2 \alpha - (\alpha-\epsilon) \log_2(\alpha-\epsilon).$$

When ϵ is such that $\alpha = 1$, the q -SC* reduces to m independent BSCs, while for $\epsilon_{i|*} = \frac{1}{2}$, $i = 1, \dots, m$, it becomes an ordinary q -SC.

The unnormalized bit APP (compare (12)) is obtained as

$$P(X_i^j = x_i^j | \mathbf{Y} = \mathbf{y}) \doteq \begin{cases} (1-\epsilon) \prod_{k=1}^m p_k^j \\ + \alpha(1-\epsilon_{i|*})p_i^j \left[\prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j) - \prod_{k \neq i} (1-\epsilon_{k|*})p_k^j \right], & x_i^j = y_i^j, \\ \alpha\epsilon_{i|*}(1-p_i^j) \prod_{k \neq i} (\epsilon_{k|*} + p_k^j - 2\epsilon_{k|*}p_k^j), & x_i^j \neq y_i^j, \end{cases}$$

leading to the LLR expression (21). The quantities p_i^j , β^j and $\beta_{[i]}^j$ are as defined in Section III. Like in the special case of the q -SC, for $\beta_{[i]}^j \rightarrow 0$ the channel LLR L_{ch} tends to a constant, which is equal to the LLR of the binary sub-channel conditioned on the occurrence of an error on one or more of the other sub-channels. The initial value of L_{ch} can again be obtained by assuming a worst-case of $p_k^j = \frac{1}{2}$, $\beta_{[i]}^j = 2^{-m+1}$ in (21), yielding

$$\begin{aligned}
L_{\text{ch}}^{(0)}(X_i^j = y_i^j) &= \log \left(\frac{1-\epsilon_{i|*}}{\epsilon_{i|*}} + \frac{1-\alpha}{\alpha\epsilon_{i|*}} \right) \\
&= \log \left(\frac{1-\alpha\epsilon_{i|*}}{\alpha\epsilon_{i|*}} \right) = \log \left(\frac{1-\epsilon_i}{\epsilon_i} \right),
\end{aligned}$$

which is the channel LLR of the marginal BSC.

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